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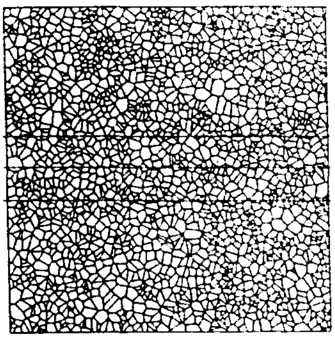

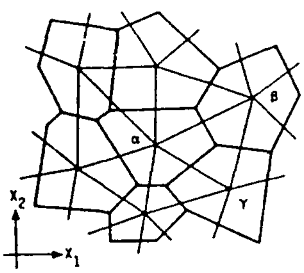

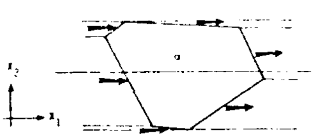

STRUCTURES LABORATORY

CONTRACT REPORT SL-92-2

PLANE WAVE PROPAGATION IN RANDOM GRANULAR MEDIA

by

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US ARMY ENGINEER WATERWAYS EXPERIMENT STATION

February 1992
Final Report

Approved For Public Release; Distribution Is Unlimited

Prepared for Defense Nuclear Agency
6801 Telegraph Road, Alexandria, Virginia 22310-3398

Under Task Code RSRB, Work Unit 00076 (Task 4)
Contract No. DACA39-89-M-1309

Monitored by Structures Laboratory
US Army Engineer Waterways Experiment Station
3909 Halls Ferry Road, Vicksburg, Mississippi 39180-6199

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. AGENCY USE ONLY (*Leave blank*)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	February 1992	Final Report

4. TITLE AND SUBTITLE

Plane Wave Propagation in Random Granular Media

5. FUNDING NUMBERS

Contract No.
DACA39-89-M-1309
Task Code RSRB
Work Unit 00076

6. AUTHOR(S)

Martin Ostoja-Starzewski

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Purdue University
West Lafayette, IN 47907

8. PERFORMING ORGANIZATION REPORT NUMBER

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

See reverse.

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

Contract Report
SL-92-2

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution is unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT (*Maximum 200 words*)

We show that a Markov property of disturbance propagation forms the basis for a study of wavefronts in graph representable microstructures. Stochastic Huygens' minor principle is developed to analyze wavefront propagation in 1-D, 2-D, and 3-D models of material microstructures. A diffusion approximation is obtained for microstructures with grains described by piecewise linear constitutive laws. A new general method of solution to transient wave problems in such microstructures is developed. The method uses solutions from deterministic problems as a reference basis for an analysis of field fluctuations and scatter in stochastic media. The wavefront is modeled as a random field in space-time governed by a Markovian propagator. Explicit formulas for random arrivals in space-time and random modulation of pulses are obtained from this diffusion approximation. The key coefficients appearing in these formulas are calculated in the dimensionless setting of a 1-D linear elastic setting for a wide range of statistics of material properties. Finally, a local averaging process is proposed to obtain various smoothing approximations of this random field.

14. SUBJECT TERMS

Granular material	Microstructure	Stochastic
Markov process	Random media	Wave propagation

15. NUMBER OF PAGES 42

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED		

9. (Continued).

Defense Nuclear Agency
6801 Telegraph Road
Alexandria, VA 22310-3398

USAE Waterways Experiment Station
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Vicksburg, MS 39180-6199

PREFACE

The investigation reported herein was conducted by Purdue University under Contract No. DACA39-89-M-1309 for the US Army Engineer Waterways Experiment Station (WES). The research was sponsored by the Defense Nuclear Agency (DNA) under Task Code RSRB, Work Unit 00076 (Task 4), "Stochastic Analysis of Wave Propagation." The research was performed during the period October 1988 through September 1989. The DNA Work Unit Manager was Dr. Edward L. Tremba, Weapons Effects Division, Shock Physics Directorate.

The principal investigation at Purdue University was performed by Professor Martin Ostoja-Starzewski. The report was written by Professor Ostoja-Starzewski, presently at Michigan State University. The work was technically monitored by Dr. Behzad Rohani, Geomechanics Division (GD), Structures Laboratory (SL), WES. Mr. Bryant Mather is Chief, SL, and Dr. John G. Jackson, Jr., is Chief, GD.

The Commander and Director of WES is COL Larry B. Fulton, EN. The Technical Director is Dr. Robert W. Whalin.

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CONVERSION FACTORS, NON-SI TO SI UNITS OF MEASUREMENT

Non-SI units of measurement used in this report can be converted to SI units as follows:

Multiply	By	To Obtain
inches	2.54	millimetres
kips (force) per square inch	6.894757	megapascals
pounds (mass) per cubic foot	16.01846	kilograms per cubic metre

I. INTRODUCTION

1.1 Review of Literature on Wavefront Propagation

One possible way of studying wave propagation is to follow the evolution of wavefronts in spacetime. A need to approach many problems in that way arises in cases of intensive dynamic loads acting on material bodies. The researches are typically based on either the theory of characteristics or the theory of propagating singular surfaces. The first of these relies on the methods of solution of hyperbolic partial differential equations and borrows heavily from developments in gas dynamics. This approach, in which the first papers appeared in the forties, developed rapidly in the sixties, and is described in the book of Cristescu (1967). This book is devoted to the transient response of continuous plastic bodies and remains a classic reference to this day. Many advances were made later, particularly in the class of one-dimensional (1-D) problems, but also in various 2-D and 3-D axisymmetric problems. Various constitutive laws were considered - such as elastic-plastic and elastic/viscoplastic - since the effort was on both solids and soils. This progress is evidenced in the monograph by Nowacki (1978). Another monograph of similar quality is the one due to Wlodarczyk (1986). We conclude this very brief account of this class of studies on wave propagation in elastic-inelastic materials with an observation that the material spatial randomness - and hence, randomness in constitutive response and its effect on wave passage - has not been accounted for.

The theory of propagating singular surfaces dates back to Hadamard (1903) who provided the celebrated Hadamard compatibility condition. This is a condition that all surfaces of discontinuity, strong or weak, have to satisfy. Evolution of surfaces of discontinuity within the framework of modern continuum mechanics was studied rigorously by Truesdell and Toupin (1960) and Truesdell and Noll (1965). In the following years this approach was utilized to examine the growth and decay behavior of shock waves and acceleration waves in various classes of materials. Many results in this

area were treated in the book by Chen (1976). Again, a reader interested in the effect of material randomness on the propagation of these discontinuity waves will come away disappointed: the deterministic continuum theory is well advanced, but the stochastic problems are not yet treated. This statement is meant not as a criticism but only as an observation; the deterministic theories always provide a basis for extensions to stochastic phenomena in physics and mechanics. Since such extensions are usually not straightforward, various existing deterministic approaches have to be learned in order to find the best route. Thus, in the remainder of this section we give short accounts of several other wavefront propagation analyses we considered.

Of special interest to us are theoretical researches on transient waves in inhomogeneous media. One of the first papers in this category was due to Sternberg and Chakravorty (1959), who investigated the propagation of rotary shear waves in an inhomogeneous isotropic elastic plate from a circular opening. The inhomogeneity was considered in the form of a power-like dependence of the shear modulus on the radial distance from the center of the hole. Indeed, this problem provided the basis for other theoretical techniques and more general assumptions. Thus, while the original problem could be tackled by a Laplace transform, a wider class of inhomogeneities was treated by a numerical scheme based on the theory of characteristics (Chou and Schaller, 1966). Following this, a new approach was developed by Achenbach and Reddy (1967) which relied on the Taylor series representation of the field quantities in the vicinity of the wavefront. This method was utilized by several authors, and was critically assessed by McCarthy (1975). In view of the drawback of the method in the setting of a deterministic medium, this line of approach does not seem promising for media with random microstructures.

Recently, a new method of solving linear wave equations has been developed by Seymour and Varley (1989). While the method can be applied to problems involving very general spatial inhomogeneities of the medium, its usefulness for us is restricted by the assumption of a linear elastic constitutive behavior.

Strong material spatial randomness of geologic media was the motivation for Rohani (1982) who initiated a program of research aimed at bringing out the microstructural nondeterministic effects. In that study the problem was considered as one of wave propagation in a homogeneous medium whose defining physical constants are random variables. In addition, the characteristics of airblast were considered random too. Several sensitivity analyses were conducted to assess the relative effects of these random quantities on the scatter of output quantities. Further studies in this vein were reported in (Rohani and Cargile, 1984).

An attempt to introduce the spatial variability of the medium's properties was undertaken in (Sadd, Hossain and Rohani, 1986). This study was based on a distributed body concept advanced by Goodman and Cowin (1972) and the associated wave propagation studies conducted by Nunziato and Walsh (1978) and Nunziato, Kennedy and Walsh (1978). The wave propagation calculations were based on the Bernoulli equation for the evolution of an acceleration wave, in which the two material coefficients were related to the microstructure in the distributed body concept. Similar as before, these coefficients were taken as space homogeneous constants assigned random values.

1.2 Motivation and Scope of the Present Research

Randomness of physical and geometrical microscale properties is an inherent characteristic of most materials. It is definitely the case with granular media, and this aspect has to be taken into account in any model aimed at a more precise description of mechanical phenomena. The phenomenon under study in this report is one of transient wave propagation in a granular microstructure; see Fig. 1a). We follow the theoretical approach of our earlier works (Ostoja-Starzewski, 1984 and 1989a). The model is developed in detail in a one dimensional (1-D) setting, that is, for a microstructure represented by a sequence of grains whose properties - mass density and constitutive moduli - are randomly assigned but constant throughout the domain of the grain. The approach adopted here is based on the recognition of the Markovian property of

disturbance propagation in so far as the ray kinematics as well as the amplitude modulation of the wave transmitted forward are concerned. Descriptions of both these processes in terms of Fokker-Planck equations are obtained, and these lead in turn to the formulas that describe the dependencies of first and second moments on the medium's statistics and the propagation times. The ensuing model is thus based on a semi-group property of a Markovian propagator, which reflects a stochastic form of Huygens' minor principle. Since this model is applicable to any characteristics of the forward propagating (transmitted) waves up to the point of intersection with another characteristic, it can be used to investigate 1-D problems of transient response of microstructures with various constitutive laws. Two cases are treated here: microstructure with linear-elastic grains, and microstructure with linear-hysteretic grains.

In the last chapter of this report we show how our theoretical approach can be extended to two and three dimensions. Starting from the Markov property of disturbance propagation we present a generalization of the diffusion approximation, which, in conjunction with solutions of transient wave problems in deterministic media, can be used to find the responses of randomly heterogeneous media. Finally, we introduce the concept of local averaging of wavefronts.

II. 1-D WAVEFRONT PROPAGATION

2.1 Basic Model

In this section we consider wavefront propagation in a 1-D model of a granular medium. That is, by a granular medium we understand a family of semi-infinite ($X_1 \geq 0$) sequences of grains; response of each sequence is independent of that of others. Each grain is taken as a homogeneous continuum of physical properties, such as mass density $\rho(\omega)$ and elastic modulus $E(\omega)$, sampled from certain probability distributions. In addition, the lengths $l(\omega)$ of grains may also be random. It is seen that a single sequence of grains is a deterministic medium $B(\omega)$, while a family $B = \{B(\omega); \omega \in \Omega\}$ is a random medium, whereby Ω denotes an underlying sample space; see also Sobczyk (1986).

We consider a space-time graph of a disturbance propagating in a class of media B in which grains are linear elastic and grain boundaries are possibly dissipative. By a disturbance we understand any single point of the pulse p(t) shown on the t-axis of Fig. 2; this pulse results in a wavefront moving in the material domain.

We observe that due to random speeds of propagation, any initial disturbance, from to an arbitrary point $p(t_o)$ of the pulse, diffuses in the space-time within a forward causality cone C^+. The cone C^+ is a set of all possible stochastic ray paths (characteristics) that originated from $\underset{\sim}{X} = X_1 = 0$ at $t = t_o$, which in view of the model's microstructure are continuous piecewise linear lines. In the following we take, without loss of generality, $t_o = 0$. Locally, the gradient of the characteristic is the phase velocity

$$c(\omega) = \left[\frac{d\sigma(\varepsilon, \omega)}{d\varepsilon} / \rho(\omega) \right]^{\frac{1}{2}} \tag{1.1}$$

Now, we introduce a dispersion distance defined by

$$\xi(t, \omega) = X(t, \omega) - \langle X(t) \rangle \tag{1.2}$$

where $<X(t)> = <c>t$ is the position of the average* path defined as a path in an average medium. It follows that $\xi(t)$ is a random walk about the origin $\xi = 0$.

A model fully equivalent to this one may be formulated by introducing a dispersion time:

$$\tau(X, \omega) = t(X, \omega) - \bar{t}(X) \tag{1.3}$$

which describes the Markovian spreading of paths about the average path, see Ostoja-Starzewski (1984). In the latter reference τ was parametrized by the average time $\bar{t} = X / <\bar{c}>$, where $<\bar{c}>$ is the ensemble average of phase velocities c of the grains.

Let us define:

$\Psi_{q_0}(t)$ — set of all materials points in B disturbed in time $\leq t$;

$\Phi_{q_0}(t) = \partial\Psi_{q_0}(t)$ — set of all material points in B reached at time t.

Thus $\Phi_{q_0}(t)$ is a disturbance at t due to a cause at point $q_0 = (X = 0, t_o = 0)$; that is, $\Phi_{q_0}(t)$ is a zone of finite rather than zero thickness as would be the case of a disturbance propagation in a deterministic medium. We note that $\Psi_{q_0}(t)$ and $\Phi_{q_0}(t)$ may also be parameterized by the average time $\bar{t}$ as indicated in Fig. 2 - then we have $\Psi_{q_0}(\bar{t})$ and $\Phi_{q_0}(\bar{t})$.

The evolution of $\Phi_{q_0}(t)$ is determined by the dynamics of the ξ_t process (definition (1.2)), or equivalently, by the $\tau_{\bar{t}}$ process (definition (1.3)). Let us consider the process ξ_t from now on. The physical properties $\rho(\omega)$ and constitutive law $\sigma(\varepsilon,\omega)$ are, in general, correlated in spatial domain. Without loss of generality we assume them to be describable by a Markov process $(\rho,\sigma)_t$. It follows then from (1.1) that ξ_t is Markov only in the sense of the vector process $(\rho,\sigma,\xi)_t$ being Markov. For simplicity, however,

* The symbol $<\cdot>$ is used to denote ensemble averages.

we shall assume henceforth that the statistics of the physical properties are spatially homogeneous in the strict sense. In that case ξ_t alone is Markov and its transition function is time-homogeneous

$$P(t,x,t+\Delta t,F) = P(\Delta t,x,F) = P\{\xi(\Delta t)\in F \mid \xi(0)=x\} \tag{1.4}$$

We thus specify this process in terms of its one-step transition function, where

$$\Delta t = \langle l/c \rangle \tag{1.5}$$

is the basis of a (yet to come) continuous diffusion process approximation of the discrete physical process. Also, F is any set in the range (state space) Ξ of the variable ξ. Furthermore, since

$$p(\Delta t,x,y) = P\ \{\xi(\Delta t)=y \mid \xi(0)=x\} = P\{\xi(\Delta t)=y-x \mid \xi(0)=0\} \tag{1.6}$$

the transition function is also homogeneous in the state space. In the above we have used a transition density

$$p(\Delta t,x,y) = \frac{\partial P(\Delta t,x,y)}{\partial y} \tag{1.7}$$

Now, we note that the above function can be determined in terms of the medium's statistics

$$p(\Delta t,x,y) = P\{\omega\text{: } X(\Delta t,\omega) - \langle c \rangle\, \Delta t = y - x\},\quad X(\Delta t,\omega) = c(\omega)\Delta t \tag{1.8}$$

The dynamics of the ξ_t process is described by a Chapman-Kolmogorov equation

$$p(t_1+t_2,\, x_0,\, x_2) = \int p(t_1,\, x_0,\, x_1)p(t_2,\, x_1,\, x_2)dx_1\ , \tag{1.9}$$

where the integration is over $C^+(t_1) \cap C^-(t_2)$, and $C^-(t_2)$ is a backward dependence cone for a time difference t_2.

It follows from the theory of Markov processes that, by introducing an operator

$$U(t)[f(x)] = \int_{\Xi} f(y)P(t, x, dy), \qquad f \in L_1(\Xi) \tag{1.10}$$

relation (1.9) leads to a semi-group property:

$$U(t_1 + t_2) = U(t_1)U(t_2) \tag{1.11}$$

which represents a stochastic form of Huygens' minor principle for the arrival of disturbance $\Phi_{q_0}(t)$.

At this stage we introduce a diffusion approximation of the ξ_t process, which is expressed by the following:

i) the Fokker-Planck equation*

$$\frac{\partial}{\partial t}p(t, x) = -\frac{\partial}{\partial x}[A_\xi(x)p(t, x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B_{\xi\xi}(x)p(t, x)] \tag{1.12}$$

in which

ii) p(t, x) is the probability density

$$p(t, x) = \int_{\Xi} p(t, x', x)dx' \tag{1.13}$$

iii) satisfying the initial condition

$$p(0, x) = \delta(x - x_o) \tag{1.14}$$

iv) and $p(t, x', x)$ being such that

---

* The time derivative here is meant in the sense of forward time differencing, i.e. $\frac{\partial}{\partial t}f(t) = \frac{f(t+\Delta t) - f(t)}{\Delta t}$, where f(t) is an arbitrary function.

$$A_\xi(x) = \frac{1}{<^\alpha\tau>} <\xi(<^\alpha\tau>) - \xi(0) \mid \xi(0) = x> \tag{1.15}$$

$$B_{\xi\xi}(x) = \frac{1}{<^\alpha\tau>} <[\xi(<^\alpha\tau>) - \xi(0)]^2 \mid \xi(0) = x> \tag{1.16}$$

$$\frac{1}{<^\alpha\tau>} <\{[\xi(<^\alpha\tau> - \xi(0)]^n \mid \xi(0) = x> \cong 0, \quad n > 2 \tag{1.17}$$

Condition (1.17) represents a constraint on strength of randomness of the medium's microstructure for the validity of the diffusion approximation.

The wave amplitude ζ evolves as a Markov process with discontinuous sample paths. In view of the spatial homogeneity of the medium's statistics, the transition function is time homogeneous and we have for a grain-grain transition

$$P(\Delta t, z, E) = P\{C : E \Rightarrow \zeta(\Delta t) = Cz\} \tag{1.18}$$

C in the above is a transmission coefficient. The Huygens' minor principle for this process is expressed by a semi-group property .

$$W(t_1 + t_2) = W(t_1)W(t_2) \tag{1.19}$$

of the operator

$$W(t)[f(z)] = \int_Z f(z)P(t, z, dz), \qquad f \in L_1(Z) \tag{1.20}$$

Also here we introduce a diffusion approximation. This is expressed by

i) the Fokker-Planck equation

$$\frac{\partial}{\partial t}p(t, z) = -\frac{\partial}{\partial z}[A_\zeta(z)p(t, z)] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[B_{\zeta\zeta}(z)p(t, z)] \tag{1.21}$$

in which

ii) p(t, z) is the probability density

$$p(t, z) = \int_Z p(t, z', z)dz' \tag{1.22}$$

iii) satisfying the initial condition

$$p(0, z) = \delta(z - z_o) \tag{1.23}$$

iv) and p(t, z′, z) being such that

$$A_\zeta(z) = \frac{1}{<^\alpha\tau>} <\zeta(<^\alpha\tau>) - \zeta(0) \mid \zeta(0) = z> \tag{1.24}$$

$$B_{\zeta\zeta}(z) = \frac{1}{<^\alpha\tau>} <[\zeta(<^\alpha\tau>) - \zeta(0)]^2 \mid \zeta(0) = z> \tag{1.25}$$

$$\frac{1}{<^\alpha\tau>} E<[\zeta(<^\alpha\tau>) - \zeta(0)]^n \mid \zeta(0) = z> \equiv 0, \quad n > 2 \tag{1.26}$$

Similar to (1.17), also (1.26) represents a constraint on the strength of randomness of the medium's microstructure. Our preliminary calculations indicate that (1.17) is more restrictive than (1.26). These calculations were based on the assumption that the random variables E and ρ have independent uniform probability densities.

We observe from (1.1) that in case of a nonlinear constitutive law, the ξ_t process is dependent on the ζ_t process. Thus, ξ_t is then Markov only in the sense of the vector process $(\zeta, \xi)_t$ being Markov. In this case we have the transition probability function

$$P(\Delta t, (z, x), E, F) = P\{\zeta(\Delta t) \in E, \xi(\Delta t) \in F \mid \zeta = z, \xi = x\} \tag{1.27}$$

2.2 Wavefront Propagation Due to an Arbitrary Pulse: Linear-Elastic Grains

For the case of a microstructure with grains whose constitutive law is piecewise linear, the ξ_t process is not explicitly dependent on the ζ_t process, providing ζ takes values in any given linear range; we consider linear-elastic grains here. Thus both processes are Markovian separately, and their transition functions are given by (1.4), and (1.15), respectively. However, their joint probability density p(t, x, z) depends on the joint transition function $p(t, x', z', x, z)$ as expressed by the conventional relation

$$p(t, x, z) = \int_{\Xi}\int_{Z} p(0, x', z')p(t, x', z', x, z)dx'dz' \tag{2.1}$$

where

$$p(t, x', z', x, z) = \frac{\partial^2 p(t, x', z', x, z)}{\partial x \partial z} \tag{2.2}$$

and $p(0, x', z')$ is the initial probability density.

We now consider the diffusion approximation for $p \equiv p(t, x,z)$. The Fokker-Planck equation reads

$$\begin{aligned}\frac{\partial}{\partial t}p = &- \frac{\partial}{\partial x}[A_\xi p] - \frac{\partial}{\partial z}[A_\zeta p]\\ &+ \frac{1}{2}\frac{\partial}{\partial x^2}[B_{\xi\xi}p] + \frac{1}{2}\frac{\partial^2}{\partial x \partial z}[B_{\xi\zeta}p] + \frac{1}{2}\frac{\partial^2}{\partial x \partial z}[B_{\zeta\xi}p] + \frac{1}{2}\frac{\partial^2}{\partial z^2}[B_{\zeta\zeta}p]\end{aligned} \tag{2.3}$$

Clearly, a complete diffusion process description of the pulse evolution is obtained with (2.3) rather than with a system of two equations (1.9) and (1.18) - this is due to the presence of "cross-terms," i.e. fourth and fifth on the right hand side above. In the following we adopt, for simplicity, the approach based on the system of (1.9) and (1.18). As a matter of fact, this corresponds to following the evolution of the pulse in terms of an operator

$$T(t) = U(t)W(t) \tag{2.4}$$

Considering first the ξ_t process we need $A_\xi(x)$ and $B_{\xi\xi}(x)$ in terms of the medium's statistics. From (1.8), (1.15) and (1.16) we obtain*

$$A_\xi(x) = 0\,, \quad B_{\xi\xi}(x) = \frac{\langle\langle c\rangle\rangle}{\langle 1/c\rangle} = \langle\langle\sqrt{\frac{E}{\rho}}\rangle\rangle\langle 1\sqrt{\frac{\rho}{E}}\rangle^{-1} = D \tag{2.5}$$

where, for simplicity of notation, we have introduced a constant D. Now, (1.12) becomes a simple diffusion equation

$$\frac{\partial}{\partial t}p(t,x) = \frac{1}{2}\, B_{\xi\xi}\, \frac{\partial^2}{\partial x^2}\, p(t,x) \tag{2.6}$$

Solution to (2.6) subject to the initial condition

$$p(t,x) = \delta(x) \quad \text{at } t{=}0 \tag{2.7}$$

is, as is well known,

$$p(t,x) = (2\pi B_{\xi\xi}t)^{-\frac{1}{2}} \exp\left[-x^2/2B_{\xi\xi}t\right] \tag{2.8}$$

Turning now to the ζ_t process, we need to determine the coefficient functions $A_\zeta(z)$ and $B_{\zeta\zeta}(z)$. These will depend on the transmission coefficient ${}^{it}C$ and the relative impedance ${}^{it}\zeta$

---

* <<·>> denotes variance

$$
{}^{it}C = \frac{2\,{}^{it}\chi}{1+{}^{it}\chi}, \quad {}^{it}\chi = \frac{{}^{t}\rho\ {}^{t}c}{{}^{i}\rho\ {}^{i}c} = \left[\frac{{}^{t}\rho\ {}^{t}E}{{}^{i}\rho\ {}^{i}E}\right]^{\frac{1}{2}} \tag{2.9}
$$

In the above, i and t denote the regions (i.e. grains) of incident and transmitted waves, respectively. From (1.18), (1.24) and (1.25) we find

$$
\begin{aligned}
A_\zeta(z) &= \frac{z}{<l/c>}\left(<{}^{it}C> - 1\right) = Az \\
B_{\zeta\zeta}(z) &= \frac{z^2}{<l/c>}\left(<{}^{it}C^2> - 2<{}^{it}C> + 1\right) = Bz^2
\end{aligned} \tag{2.10}
$$

where, for simplicity of notation, we have introduced two constants A and B. It follows that the Fokker-Planck equation (1.21) is, in Van Kampen's (1981) terminology, a so-called nonlinear equation; the cause of the nonlinearity lies in the multiplicative character of the process ζ_t as reflected by the transition function (1.18). At this stage we can either seek the solution p(t,z), or assess the time dependence of the first and second moments only.* We decide here on the latter alternative, and hence obtain from (1.21)

$$
\begin{aligned}
\frac{d}{dt}<\zeta> &= <A_\zeta(z)> \\
\frac{d}{dt}\ll\zeta\gg &= 2<(\zeta - <\zeta>)A_\zeta(z)> + <B_{\zeta\zeta}(z)>
\end{aligned} \tag{2.11}
$$

These have to be supplemented with the initial conditions

---

* Solution of the equation (2.3) is obtained in Ostoja-Starzewski (1991).

$$<\zeta> = z_o \text{ and } \ll\zeta\gg = 0 \text{ at } t=0 \tag{2.12}$$

In view of (2.10), relations (2.11) become

$$\begin{aligned} &\frac{d}{dt}<\zeta> = A<\zeta> \\ &\frac{d}{dt}\ll\zeta\gg = (2A + B)\ll\zeta\gg + B <\zeta>^2 \end{aligned} \tag{2.13}$$

Finally, we obtain (note from (2.12) that $\ll\zeta(0)\gg = 0$)

$$\begin{aligned} &<\zeta(t)> = z_o \exp (At) \\ &\ll\zeta(t)\gg = \ll\zeta(0)\gg \exp [(2A + B)t] + \int_0^t <\zeta(t)>^2 B \exp [(2A + B) (t - s)] \, ds \end{aligned} \tag{2.14}$$

In this section we study wavefront propagation in a microstructure made of linear-elastic grains, occupying the $X \geq 0$ half-space, and subjected to a surface pressure f(t) at $X = 0$, see Fig. 2. The constitutive law of any grain is

$$\sigma = E(\omega)\varepsilon \tag{2.15}$$

where ω indicates that the Young's modulus E is a random variable. The above statement corresponds to a problem of wave propagation in a 1-D rod made of elastic elements with varying moduli. By introducing the constitutive law

$$\sigma = [\lambda(\omega) + 2\mu(\omega)]\varepsilon \tag{2.16}$$

we have a problem of wave propagation in a half-space made of layers of varying (random) moduli $\lambda + 2\mu$. For simplicity of notation we stay with equation (2.15).

There are two regions, I and II, in the space-time domain (Fig. 3). In region I we have for stress, strain, and particle velocity:

$$\sigma_I = 0, \quad \varepsilon_I = 0, \quad v_I = 0 \tag{2.17}$$

For any specific medium $B(\omega)$, region I is bounded by a trajectory above which the body is disturbed; the latter region is denoted by II. The stress, strain, and particle velocity in region II are determined jointly by the ζ_t-process, the constitutive law (2.15) and the well-known relation

$$v = -c\varepsilon \tag{2.18}$$

If the pressure is prescribed it is most natural to represent stress by ζ. In view of the developments of section 2.1, we obtain a family of boundaries $X(0, \omega)$ separating both regions, that is, undisturbed I from disturbed II. Rather than working with this "fuzzy" ever-broadening boundary we introduce the mean path OA and trace the character of evolution of the disturbance in the random medium $\mathcal{B} = \{B(\omega); \omega \in \Omega\}$ in terms of the process referred to this path.

Now, it follows by the argument of linearity of response of all the grains (see equation (2.15)) that the propagation speed $c(\omega)$ is same for every stress level in any given grain. Hence, for any fixed time t, the trajectory $X(t, \omega)$ is same as $X(0, \omega)$ for any given body $B(\omega)$, except for the time shift t. This is analogous to two characteristics being parallel in case of wave propagation in a homogeneous deterministic microstructure. In fact, it follows that the reference paths are parallel for a linear elastic microstructure.

2.3 Wavefront Propagation Due to a Square Pulse: Linear-Hysteretic Grains

In this section we study wave propagation in a body whose elements (i.e. grains in case of a rod model, or layers in case of a half-space model) are governed by a stress-strain diagram of Fig. 4. The stress-strain curve is a straight line OM on initial loading to M; its slope defines the initial modulus E_0. Upon unloading the stress-strain curve is another straight line MN which defines the second modulus E_1

$$\sigma - \sigma_m = E_1(\varepsilon - \varepsilon_m) \tag{3.1}$$

If material is reloaded, it follows the line NM to M, and then continues along the initial loading line. E_o and E_1 are random from grain to grain; they correspond to phase velocities c_0 and c_1, respectively.

We consider wave propagation under action of a pressure pulse f(t) applied at $X = 0$, where f(t) is a square pulse

$$f(t) = \begin{cases} p_o & \text{for} \quad 0 \le t < t_1 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

The problem with this initial condition for a deterministic homogeneous medium, which was solved by Salvadori et al (1960) (see also Rohani, 1970), forms the reference basis for solution of a stochastic problem. In Fig. 5 we give the space-time diagram of the deterministic problem. We see that there are several regions: I, II, III, IV, V, ... Region I is that of an undisturbed body, i.e.

$$\sigma_I = 0, \ \ \varepsilon_I = 0, \ \ v_I = 0 \tag{3.3}$$

Region II corresponds to a material in which

$$\sigma_{II} = -p_o, \ \varepsilon_{II} = \sigma_{II}/E_o, \ v_{II} = -c_o\varepsilon_{II} \tag{3.4}$$

the boundary between I and II being given by

$$X = c_o t \tag{3.5}$$

In region III we have

$$\sigma_{III} = 0, \ v_{III} = v_I - \frac{p_0}{\rho c_1} \tag{3.6}$$

The time t_2 at which the pressure p, reaches the original wavefront is obtained as

$$t_2 = \frac{c_1}{c_1 - c_o} t_1 \tag{3.7}$$

As discussed in the above reference, at time t_2 there is a reflection of the wavefront travelling at velocity c_1 (i.e. upper boundary of region II) from the original wavefront travelling at c_o so that a new region IV is created. The field quantities in IV are

$$\sigma_{IV} = \alpha p_0, \; v_{IV} = -\sigma_{IV}/\rho c_1 \tag{3.8}$$

where

$$\alpha = \frac{c_1/c_o - 1}{c_1/c_o + 1} \tag{3.9}$$

The analysis can be continued in this fashion to cover the entire X, t-plane, and to study arbitrary pulses f(t).

Turning now to the random medium problem we see that each of the lines in Fig. 5 representing the discontinuity waves (i.e. shocks) can be considered as a mean path providing reference for stochastic processes ξ and ζ. Thus, for example, the leading shock is a reference for a family of characteristics

$$X(0, \omega, +c_o) = X(t, \omega, +c_o)\,|_{t=0}, \; \omega \in \Omega \tag{3.10}$$

Similarly, the line bounding the region II from above is a family of characteristics

$$X(t_1, \omega, +c_1) = X(t, \omega, +c_1)\,|_{t=t_1}, \; \omega \in \Omega \tag{3.11}$$

and so forth. The cones corresponding to the ξ processes along the above mentioned two mean characteristics (paths) are shown in Fig. 6. Clearly, the point of intersection will be diffused about the reference point ($X = \langle c_o \rangle t_2$, t_2).

First, we consider one characteristic $X(0, \omega, +c_o)$ which is slower than the mean one. This, we may say, corresponds to an effective velocity $c_o = c_{o,ef}$ for body $B(\omega)$ being smaller than $\langle c_o \rangle$; $\langle c_o \rangle$ defines the average path according to (1.2). In case the

moduli $E_o(\omega)$ and $E_1(\omega)$ in the constitutive law are independent random variables, there is no way to predict whether the characteristic $X(t_1, \omega, +c_1)$ for the same body $B(\omega)$ will also be slower than the mean one corresponding to $<+c_1>$. In that case the intersection of both characteristics - $X(0, \omega, +c_o)$ and $X(t_1, \omega, +c_1)$ - may take place anywhere in the quadrilateral ABFE. In the special case of dependence between random variables $E_o(\omega)$ and $E_1(\omega)$ being such that

$$E_o(\omega) < <E_0> \text{ if and only if } E_1(\omega) < <E_1> \tag{3.12}$$

and the same holding for ">" signs in place of "<", the point of intersection will fall in the HBFO$'$ quadrilateral.

Now, if we consider a characteristic $X(0, \omega, +c_o)$ which is faster than the mean one, we obtain immediately the following conclusions in a complete analogy to the above analysis:

i) for no dependence of moduli $E_o(\omega)$ and $E_1(\omega)$, the intersection will occur somewhere in the quadrilateral EDCF,

ii) for a dependence of moduli according to (3.12), the intersection will occur somewhere in the quadrilateral EDGO$'$.

It is seen from the geometry of the problem that, in case of dependence between $E_o(\omega)$ and $E_1(\omega)$ for all ω, the scatter in point of intersection of two characteristics - as measured by the distance between points AC - increases for the ratio E_1/E_o decreasing to 1. This indicates that even a weak randomness in the medium's properties may alter certain aspects of its response in a significant way.

On the other hand, in case of dependence defined by (3.12), the scatter in the location of point of intersection of two characteristics, as measured by the distance between points B and D, is much weaker.

The foregoing analysis of the intersection of characteristics for a random microstructure governed by a bilinear-type constitutive law carries over to any

intersection point of Fig. 5, as well as to all such points of other bilinear-type (and piecewise-linear) problems.

2.4 Numerical Results

In Section 2.2 we have derived the diffusion equations governing the behavior of probability densities p(t,x) and p(t,z). From these we have obtained explicit formulas for p(t,x) and the first and second moments of the variable ζ, that is equations (2.8) and (2.14). These formulas are governed by the coefficients D, A, and B, which, in turn, depend on the three random variables l, ρ, and E describing the microstructure. It is of interest now to investigate the effects of this dependence.

Consonant with our assumption of a strict spatial homogeneity of material properties (recall the discussion preceding equation (1.4)) we describe the properties of any grain by a density $p(l,\rho,E)$, and, in what follows, assume a particular special case

$$p(l,\rho,E) = p(l)p(\rho)p(E) \tag{4.1}$$

Calculations of D, A and B according to $(2.5)_2$ and (2.10) are unwieldy. Indeed, even the calculation of the probability density of the phase velocity $c = \sqrt{E/\rho}$ is lengthy. The problem qualifies perfectly for an application of a Monte Carlo technique. Results based on this approach are summarized in Tables 1 and 2; C denotes <c> here. These calculations correspond to the situation of p(l), p(ρ), and p(E) being uniform densities with means equal 1 and standard deviations σ_l, σ_ρ, and σ_E ranging from 0.02 to 0.2. In order to bring out the relative importance of randomness of l versus ρ versus E, four general cases were considered:

i) $\sigma_l = \sigma_\rho = \sigma_E = 0.02$ through 0.2: left part of Table 1,

ii) $\sigma_l = 0, \sigma_\rho = \sigma_E = 0.02$ through 0.2: right part of Table 1,

iii) $\sigma_l = \sigma_E = 0$, $\sigma_\rho = 0.02$ through 0.2: left part of Table 2,

iv) $\sigma_l = \sigma_\rho = 0$, $\sigma_E = 0.02$ through 0.2: right part of Table 2.

TABLE 1

$\sigma_l=\sigma_\rho=\sigma_E$	A	B	C	D	$\sigma_l=0$ $\sigma_\rho=\sigma_E$	A	B	C	D
0.02	-.0007	.0005	1.0002	.0002	0.02	-.00005	.0005	1.0001	.0002
0.04	-.0008	.0002	1.0003	.0008	0.04	-.0005	.0002	1.0003	.0008
0.06	-.0009	.0005	1.0009	.0019	0.06	-.0009	.0005	1.0010	.0018
0.08	-.0014	.0008	1.0016	.0033	0.08	-.0014	.0008	1.0020	.0032
0.10	-.0020	.0013	1.0024	.0050	0.10	-.0019	.0012	1.0023	.0052
0.12	-.0044	.0019	1.0043	.0075	0.12	-.0029	.0018	1.0026	.0072
0.14	-.0053	.0025	1.0049	.0102	0.14	-.0045	.0025	1.0044	.0100
0.16	-.0075	.0033	1.0095	.0135	0.16	-.0066	.0032	1.0078	.0134
0.18	-.0097	.0043	1.0076	.0171	0.18	-.0091	.0043	1.0096	.0173
0.20	-.0106	.0054	1.0092	.0214	0.20	-.0102	.0052	1.0132	.0219

TABLE 2

$\sigma_l=\sigma_E=0$ σ_ρ	A	B	C	D	$\sigma_l=\sigma_\rho=0$ σ_E	A	B	C	D
0.02	-.0000	.0000	1.0001	.0001	0.02	-.0000	.0000	1.0000	.0000
0.04	-.0002	.0001	1.0006	.0004	0.04	-.0002	.0001	.9998	.0004
0.06	-.0004	.0002	1.0013	.0009	0.06	-.0004	.0002	.9996	.0009
0.08	-.0011	.0004	1.0030	.0016	0.08	-.0011	.0004	.9986	.0016
0.10	-.0012	.0006	1.0038	.0026	0.10	-.0013	.0006	.9988	.0025
0.12	-.0018	.0009	1.0054	.0037	0.12	-.0018	.0009	.9983	.0037
0.14	-.0024	.0013	1.0077	.0052	0.14	-.0025	.0013	.9978	.0050
0.16	-.0028	.0017	1.0089	.0068	0.16	-.0027	.0017	.9974	.0065
0.18	-.0036	.0021	1.0117	.0089	0.18	-.0036	.0021	.9969	.0083
0.20	-.0047	.0027	1.0149	.0113	0.20	-.0046	.0027	.9959	.0104

The numbers of both tables were obtained by using 10^4 samples for each specific case (i.e. each specification of all three σ's). The last digits are only approximate since a rather simple random number generator was employed.

The results of Table 1 show that the effect of random grain lengths vis-à-vis the deterministic ones is practically negligible. On the other hand, restricting the material randomness to ρ alone, or E alone, has a definite effect of decreasing the randomness of the system: weaker attenuation and smaller diffusion. Interestingly, case iv) is the only one which results in $<c> < 1$.

The above being a dimensionless formulation it clearly offers a possibility of calculating any particular dimensional case if the means μ and standard deviations σ of l, ρ, and E are given. Thus, for example, if the following case is given:

$$\mu_l = 1'' = 0.0254\text{m} \qquad , \sigma_l = 0.0$$

$$\mu_\rho = 100\ \text{lb/ft}^3 = 1602\ \frac{\text{kg}}{\text{m}^3}, \quad \sigma_\rho = 0.1 \cdot \mu_\rho \tag{4.2}$$

$$\mu_E = 63\ \text{ksi} = 3016440 \frac{\text{N}}{\text{m}^2}, \quad \sigma_E = 0.1 \cdot \mu_E$$

we read from the right part of Table 1 the following values:

$$A = -0.0019,\ B = 0.0012,\ C = 1.0023,\ D = 0.0052 \tag{4.3}$$

Now, the dimensional values of the four constants - denoted by $\check{A}$, $\check{B}$, $\check{C}$, and $\check{D}$ - can be obtained from these formulas

$$\check{A} = \left[\mu_l \sqrt{\frac{\mu_\rho}{\mu_E}}\right]^{-1} \cdot A$$

$$\check{B} = \left[\mu_l \sqrt{\frac{\mu_\rho}{\mu_E}}\right]^{-1} \cdot B$$

$$\check{C} = \sqrt{\frac{\mu_E}{\mu_p}} \cdot C \tag{4.4}$$

$$\check{D} = \frac{\mu_E}{\mu_p} \left[\mu_1 \sqrt{\frac{\mu_p}{\mu_E}} \right]^{-1} \cdot D$$

Thus, (4.3) yields (in the SI system)

$$\check{A} = -3.25 \frac{1}{s} \qquad \check{B} = 2.05 \frac{1}{s}$$

$$\tag{4.5}$$

$$\check{C} = 43.49 \frac{m}{s} \qquad \check{D} = 9.79 \frac{m^2}{s^2}$$

III. FRAMEWORK FOR THE WAVEFRONT PROPAGATION STUDIES IN 2-D AND 3-D RANDOM MICROSTRUCTURES

3.1 Basic Model

The stochastic model of disturbance evolution presented in the preceding section forms a basis for a treatment of wavefronts propagating in 2-D and 3-D media. For illustration of the analysis, we consider the 2-D case here only. Specifically, we look at microstructures representable by graphs, such as the one shown in Fig. 1a). As discussed by Ostoja-Starzewski (1987, 1989a), thin lines connect the centers (vertices of the set V) of interacting grains and form set E of graph G(V, E). On the other hand, thick lines of the set E' outline the contours of polygon-shaped grains; they meet in triplets at vertices of the set V', and hence we have the graph $G(V', E')$. It is assumed that the space of any polygon domain is filled by a homogeneous continuum so that there are no voids.* The continuum itself is, in general, anisotropic and hence the propagation characteristics of any type of wave (e.g., *dilatational) are given by an envelope of velocity vectors*, i.e. an indicatrix, specific to any grain. Thus, the microstructure can be characterized by a set of indicatrices, each of which is attached at the vertex of G(V, E), see Fig. 1b).

The microstructure represented in Fig. 1a) and 1b) is just one possible realization of microscale geometrical and physical properties of the medium in this part of spatial domain, and therefore it is a deterministic medium $B(\omega)$; ω indicates a parameterization by an element from the sample space Ω. Accordingly, a random medium B is a family $\{B(\omega);\ \omega \in \Omega\}$ of all bodies thus described, with probability distributions of microstructural geometries and physical characteristics specified. In the following we

* This model permits a simulation of actual voids (i.e. absence of grains) by letting the elastic moduli of a given polygon domain be equal to zero.

assume these probability distributions to be spatially homogeneous.

Evolution of disturbances follows the kinematics of rays governed by two equations

$$\begin{aligned} d\underset{\sim}{X} &= c^2 \underset{\sim}{Y} dt \\ d\underset{\sim}{Y} &= c \underset{\sim}{\nabla} \left[\frac{1}{c} \right] dt \end{aligned} \tag{1.1}$$

where $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are the position and direction, respectively, of the ray. Evidently, the pair $(\underset{\sim}{X}, \underset{\sim}{Y})$ undergoes a Markovian evolution, and we can write its transition function in terms of the medium's statistics. However, it is more convenient to formulate this process in terms of the dispersion vector (recall (1.2))

$$\underset{\sim}{\xi}(t, \omega) = \underset{\sim}{X}(t, \omega) - \langle \underset{\sim}{X}(t) \rangle \tag{1.2}$$

and the angle η shown in Fig. 7. In this figure we give a space-time graph of plane disturbance propagation in X_1, X_2-space, where the initial conditions are $\underset{\sim}{X}(0) = (0, 0)$, $\underset{\sim}{Y}(0) = \frac{1}{c}(1, 0)$.

Introducing the transition probability function

$$P(t, \underset{\sim}{x}, y, E) = P\{[\underset{\sim}{\xi}(t), \eta(t)] \in E \mid \underset{\sim}{\xi}(0) = \underset{\sim}{x}, \eta(0) = y\} \tag{1.3}$$

we have a Markovian operator

$$U(t)[f(\underset{\sim}{x}', y')] = \int_{\Xi} \int_{H} f(\underset{\sim}{x}, y) P(t, \underset{\sim}{x}', y', dx', dy'), \qquad f \in L_1(\Xi \times H) \tag{1.4}$$

which satisfies the semi-group property

$$U(t_1 + t_2) = U(t_1) U(t_2) \tag{1.5}$$

Similar to the 1-D model, the spatial inhomogeneity of the medium has, besides randomization of characteristics, an effect on wave amplitude carried by any disturbance. In Fig. 8a) we show a typical grain α of the microstructure in which a plane disturbance propagates. The modulation of the disturbance amplitude in its passage through grain α is described by a transmission operator

$$C : {}^{\alpha}\underset{\sim}{\zeta}^{i}(t) \rightarrow {}^{\alpha}\underset{\sim}{\zeta}^{t}(t + \Delta t) \tag{1.6}$$

Here, again, $\underset{\sim}{\zeta}$ is a field quantity such as stress or strain, and i and t indicate the incident and transmitted quantities, respectively. Evolution of the disturbance takes place along the actual random ray path specific to the grain α; the average path is shown for reference. The development at the end of Section 2.1 applies here as well, and thus W(t) is the appropriate evolution operator.

3.2 Spherical and Cylindrical Waves, and Local Averaging

The approach presented in the 1-D setting (Section II) may be briefly described as a combination of the method of characteristics with the diffusion process approximation of arrival times and field quantities referred to the mean paths (i.e. average characteristics). This suggests a method for 2-D and 3-D settings: use the solutions to deterministic wave problems with spherical or cylindrical symmetry as a reference for the diffusion processes $(\underset{\sim}{\xi},\eta)_t$ and $\underset{\sim}{\zeta}_t$ in order to assess the results (e.g. strength of fluctuations) in random media (books by Nowacki (1978) and Wlodarczyk (1986) are good references on such problems). In these problems, on account of the assumed symmetry, all the field quantities are functions of time and a single spatial variable (radius r). However, higher level of complexity is encountered here due to the presence of more complex stress states than those in the 1-D problems. Of course, any type of spatial symmetry is not a condition for the applicability of our Markovian theory, but then more complicated analyses need to be carried out.

Finally, we note that the entire approach adopted here gives descriptions of the wavefront $\Phi_{q_0}(t)$ - random field in space-time - with the scale of resolution equal to the average grain size <l>. Evolution of $\Phi_{q_0}(t)$ is described along any given average characteristic by the propagator T(t) such that (Huygens' minor principle)

$$T(t_1 + t_2) = T(t_1)\, T(t_2) \tag{1.7}$$

This corresponds, in fact, to a local averaging of the true wave process at the scale <l>. By using the notion of local averaging of random fields (Vanmarcke, 1983) we can arrive at less detailed but smoother descriptions of $\Phi(t)$, all of them parameterized by $\delta = L/\langle l \rangle$, where L is the scale of averaging. The situation is akin to the notion of a window in the problem of continuous parameter random field approximations in quasi-statics of random microstructures; see Ostoja-Starzewski and Wang (1989a and b). In the latter case L is the edge length of a window ΔV, ΔV being a representative volume element. Clearly, for any $\delta < \infty$ we have a statistical continuum, while for $\delta \to \infty$ we obtain a deterministic continuum since the number of microelements (e.g. grains) in ΔV becomes infinite. Returning to our present problem we show a ray tube of width L or δ in the microstructure of Fig. 8b). Hence, the analysis can be conducted in terms of a transmission operator $C(\delta)$ and a propagator $T(t, \delta)$ corresponding to a given approximation δ.

IV. CONCLUSIONS

In this report we outlined a model for analysis of stress wavefronts propagating in granular media without voids. The following are two keystones of the model:

— microstructure is representable by a graph,

— attention is focused on the evolution of a single point of wavefront (pulse) called a disturbance.

The report is based on our previous work on wave propagation in discrete random media. Specifically, we use the result that the evolution of a forward propagating disturbance, travelling along its stochastic characteristic, is described by a random vector process $(\zeta,\xi)_t$; ζ is the amplitude of the field quantity (e.g. stress) at the disturbance, and ξ is the fluctuation in distance covered by the disturbance up to time t. Thus, with ζ we model stochastic temporal variability of the amplitude of, say, stress, while with ξ we model diffusion of characteristics in space-time, and hence, scatter in the arrival times.

$(\zeta,\xi)_t$ is a Markov process for a microstructure with grains of a nonlinear type, where nonlinearity reflects either a non-Hookean nature of elastic grains, or a nonelastic nature (e.g. elastic-plastic) of their constitutive laws. In the case of a nonlinear constitutive law there arises an interesting phenomenon of curving of the effective average paths of forward propagating waves; this is presently under study and will be reported in the next paper. It is important to note that, $(\zeta,\xi)_t$ is, strictly speaking, Markov only for grains whose properties are not mutually dependent in space, i.e. when the physical properties of the grains (mass density and constitutive moduli) are independent random variables. Otherwise, the process $(\zeta,\xi)_t$ retains its Markov property only in the sense of being driven by the random Markov process of physical properties.

A very practical approximation of the $(\zeta,\xi)_t$ process is obtained for grains whose constitutive law is piecewise linear. The simplest approximation is obtained in the form of two Fokker-Planck equations - one for ζ_t and one for ξ_t - in which the key coefficient

functions can be calculated explicitly in terms of the statistics of medium's properties. The ξ_t process is then governed by a linear diffusion equation, and hence its probability density at any arbitrary time $t > 0$ is Gaussian. The ζ_t process is a multiplicative one - its solution is found here in terms of the first and second moments only; complete solution for the probability density is possible too. The rates of change of field quantities in the model are referred to the temporal microscale in the problem: $\Delta t = \langle l/c \rangle$.

After discussing the implications of the above stochastic model for a microstructure of linear elastic grains, we turn to the case of linear-hysteretic grains. We develop a method of solution, also applicable to all transient wave problems with microstructures of piecewise linear constitutive laws, which uses the space-time diagram of the deterministic medium problem as a reference for the average paths of the stochastic setting. It is shown that there can occur a very strong scatter in solutions due to an even small randomness of the medium.

In the presentation of our recent conference paper (Ostoja-Starzewski, 1989b) we have discussed an extension of this analysis to microstructures governed by bilinear-elastic and nonlinear elastic laws with random constitutive coefficients. In the first case there is a possibility of an abrupt change in the orientation of the forward characteristics, while in the second case there is a continuous curving of characteristics; see (Ostoja-Starzewski, 1991) for analysis of interesting phenomena which arise in the first situation.

In a separate section of the report we present results of calculation of the key coefficients (drift and diffusion) which appear in the equations governing the evolution of forward characteristics and modulation of pulse strength. The calculations are carried out in the dimensionless setting of a 1-D linear elastic model. This has an obvious advantage that any physical (dimensional) case can be calculated with the formulas (4.4) and Tables 1 and 2, under the condition that the actual case satisfies the assumptions of statistical independence of generic random variables l, ρ, and E describing the microstructure.

In the final part of the report we outline a framework for solving 2-D and 3-D problems. First, we present a generalization of the disturbance propagation process in terms of vector Markov processes. Analogous to the 1-D setting, this can be combined with the classical solutions of deterministic problems -- typically well known in cases of spherical and cylindrical symmetry -- to obtain solutions of stochastic problems. Finally, we briefly introduce the concept of local averaging of wavefronts -- an idea which should be very useful in matching experimental measurements with this theory.

It is our opinion that a model of transient wave propagation in randomly heterogeneous materials has to be derived from micromechanical considerations. The key observation is that a propagating disturbance *is affected* by the random microstructure of the material. Hence, one has to be very careful with applying effective continuum models that were initially derived for quasistatic situations. As the analysis in this report shows the random nature of micro-scale properties, even with a weak strength of fluctuations, may result in rather strong macroscopic effects.

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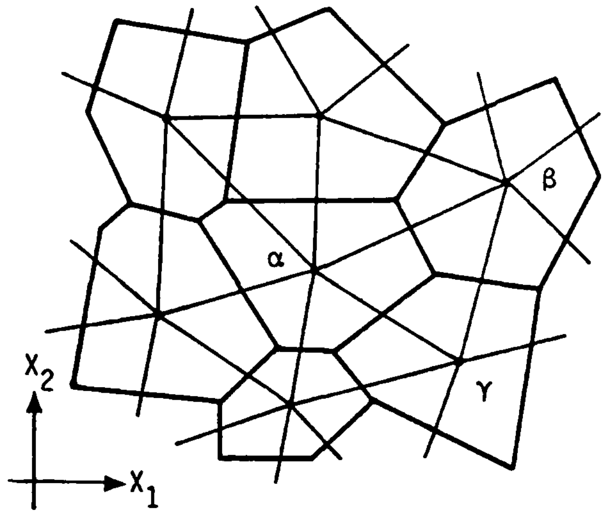
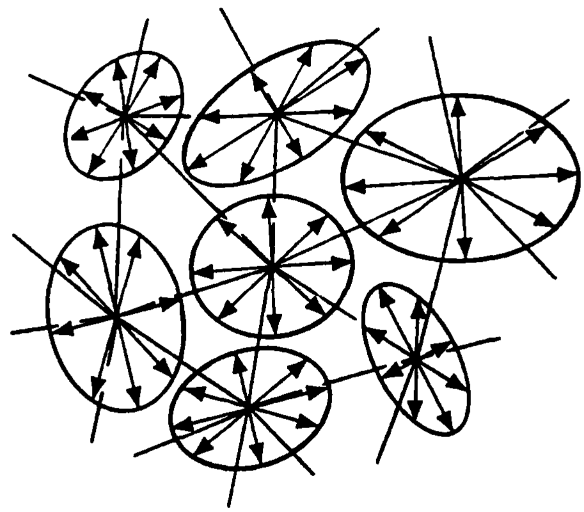


a. Graph representation of the microstructure of a heterogeneous solid



b. Distribution of indicatrix envelopes for the same ensemble of grains

Figure 1

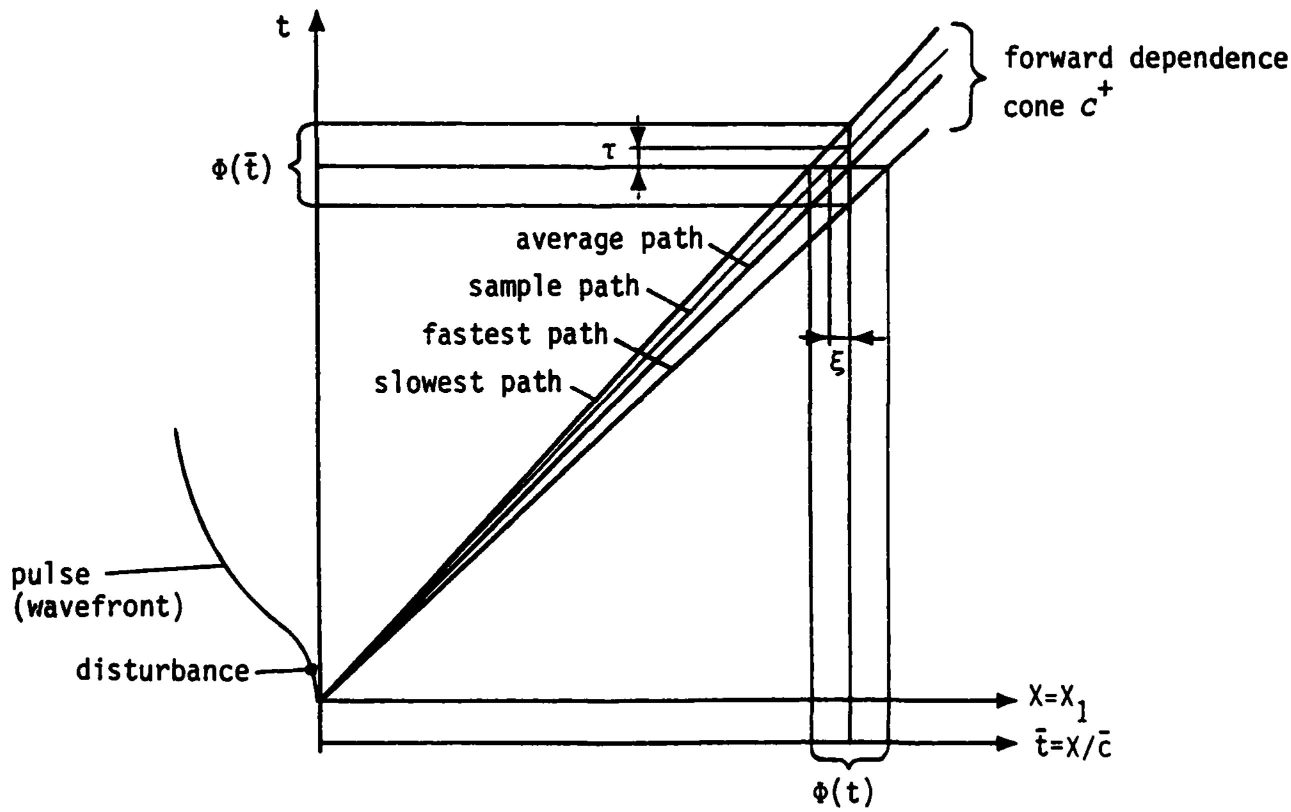


Figure 2. Space-time graph of 1-D distribution propagation

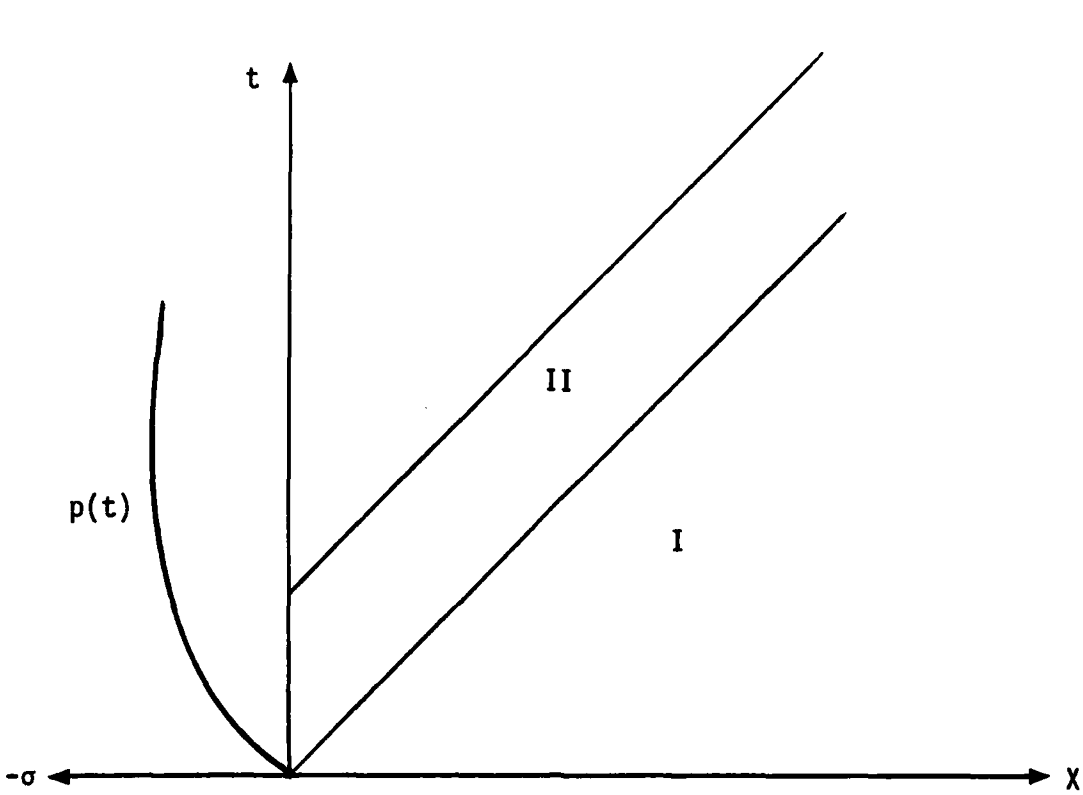


Figure 3. Space-time graph of response of a linear-elastic medium

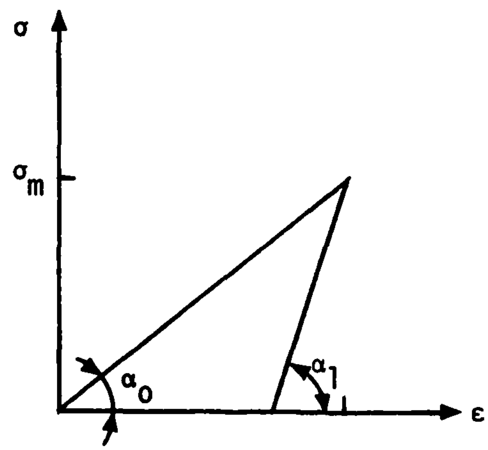


Figure 4. Linear-hysteretic constitutive law

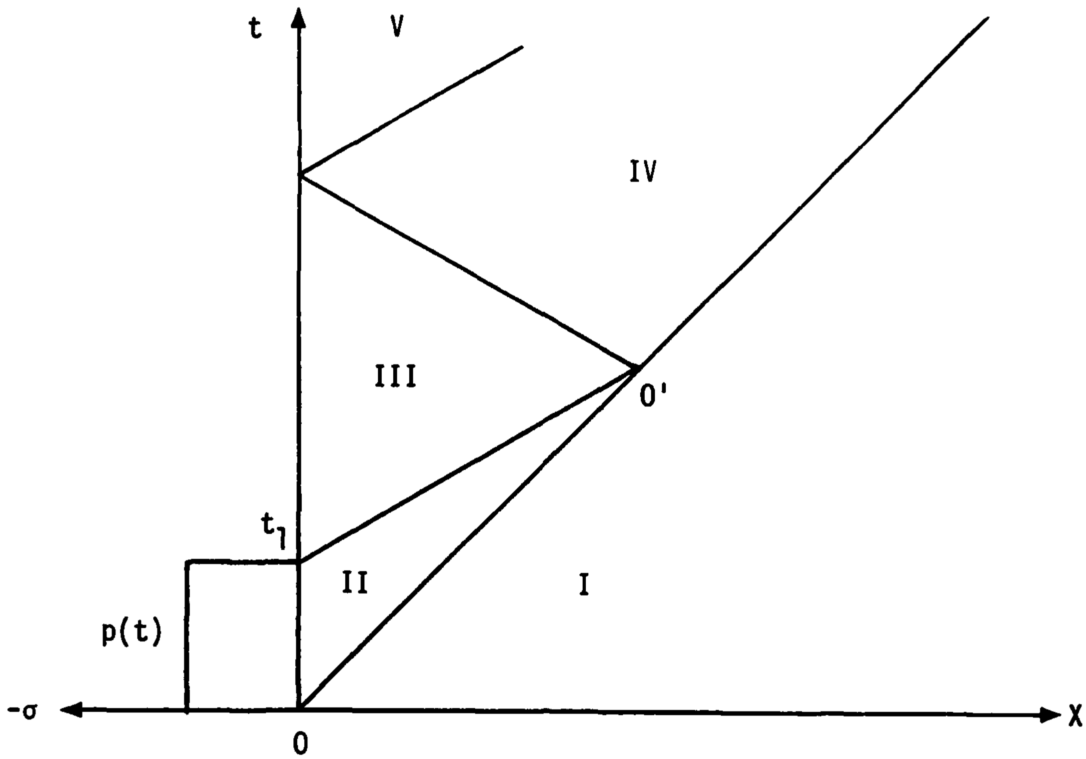


Figure 5. Space-time graph of response of a linear-hysteretic medium to a square pulse

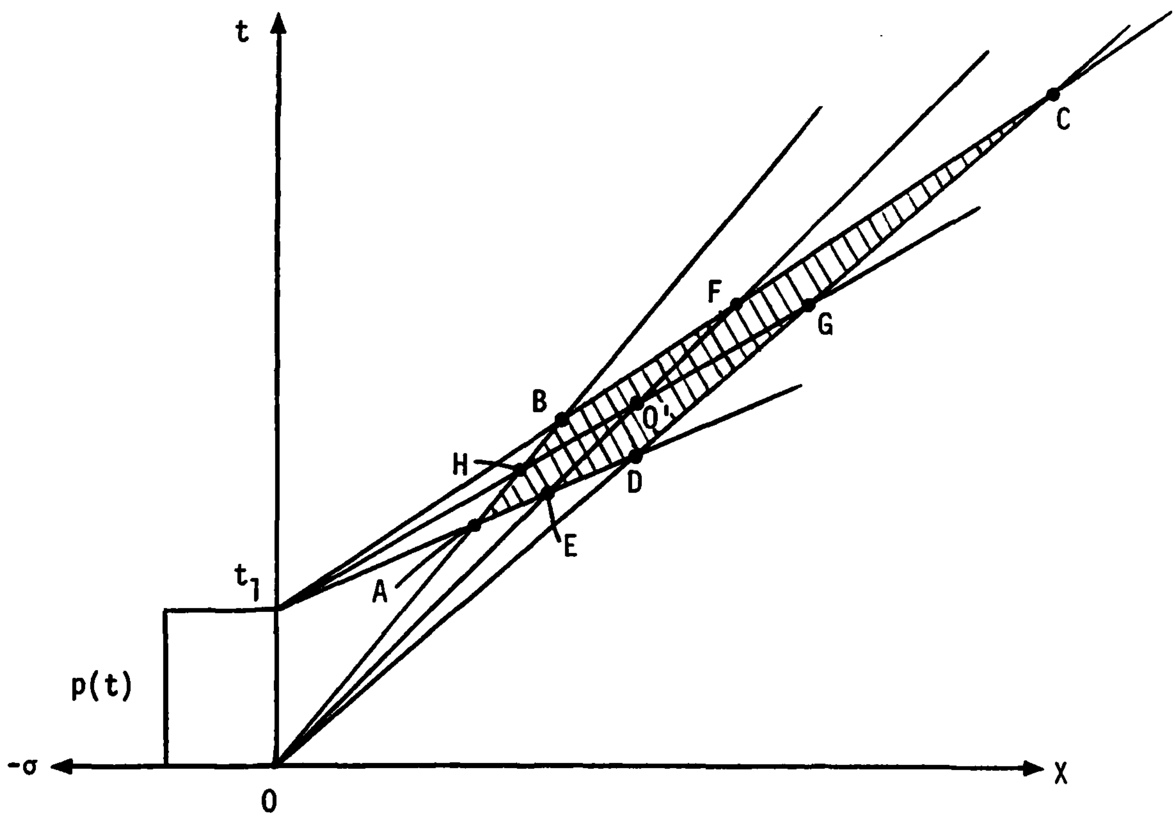


Figure 6. Intersection of forward dependence cones in response of a random linear-hysteretic medium to a square pulse

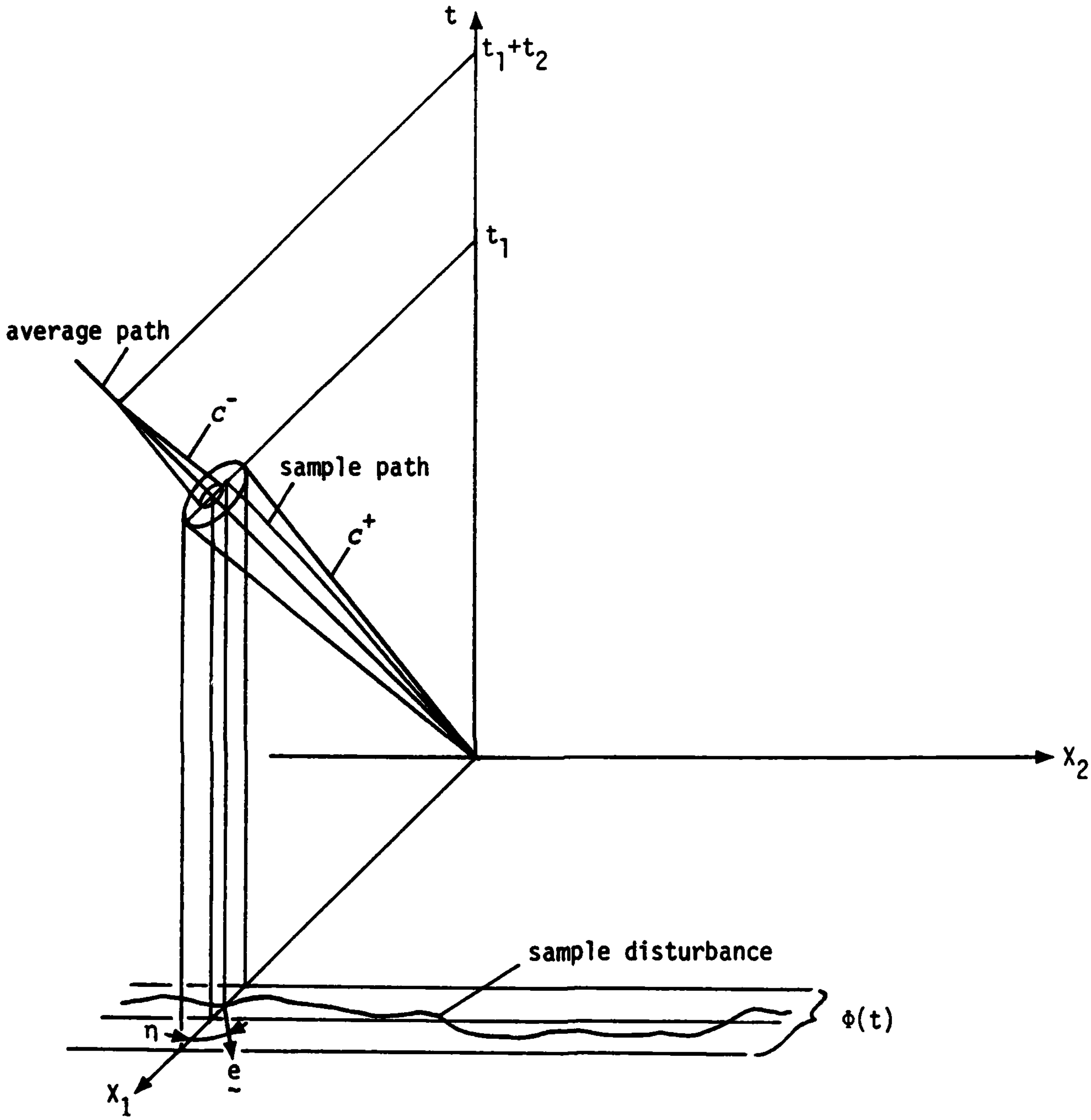
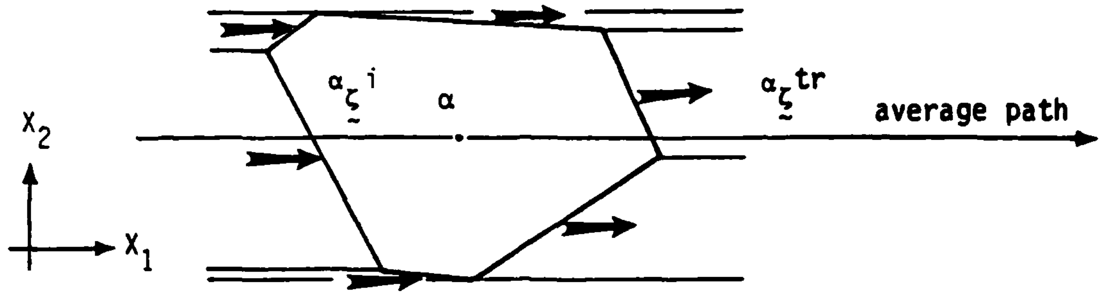
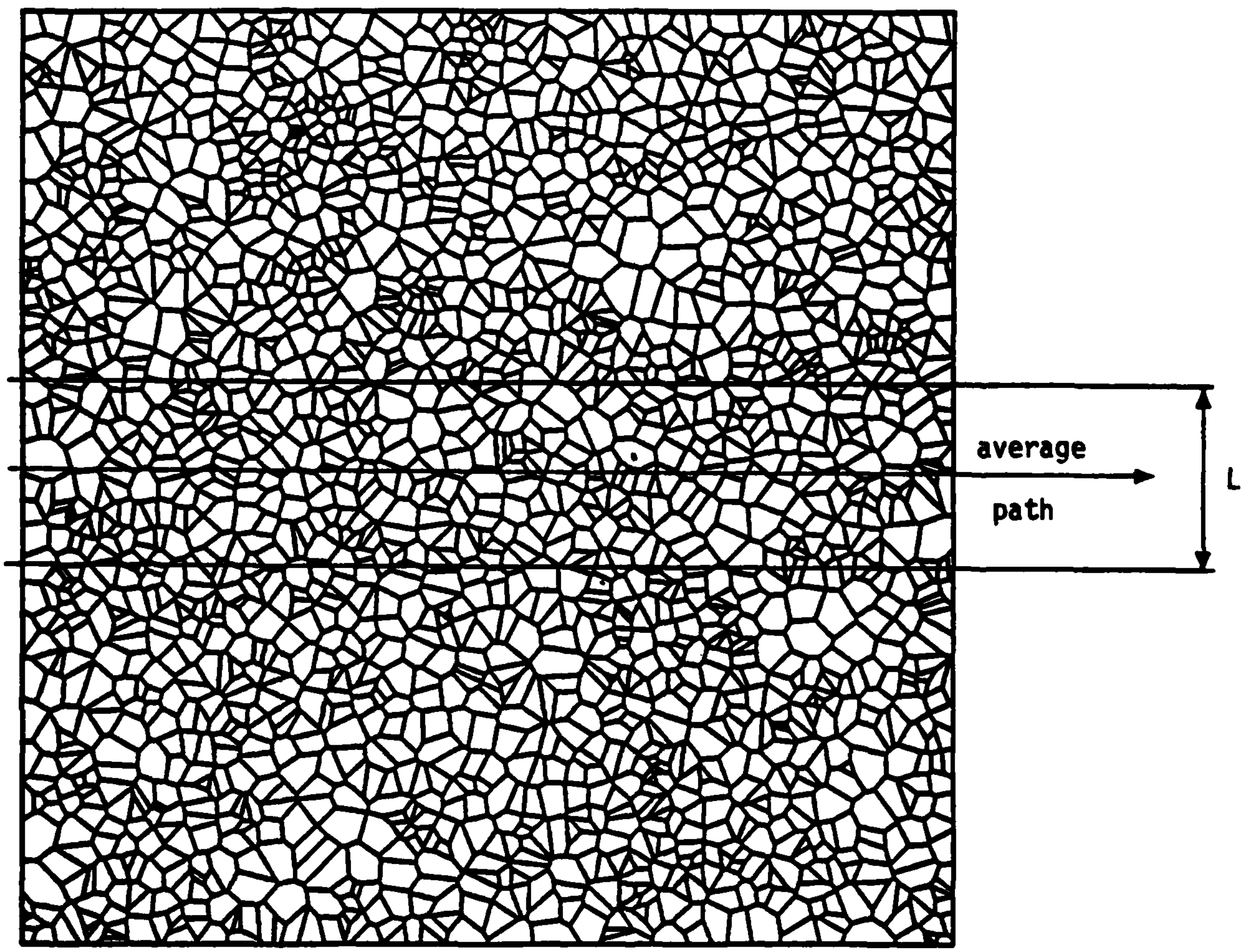


Figure 7. Space-time graph of 2-D distribution propagation



a. Wave passage through a grain



b. Wave passage through a microstructure

Figure 8

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